

CURVATURE FORMULAS OF EXTENDED HOLOMORPHIC CURVES ON C^* -ALGEBRAS AND SIMILARITY OF COWEN-DOUGLAS OPERATORS

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ABSTRACT. For $\Omega \subseteq \mathbb{C}$ a connected open set, and \mathcal{U} a unital C^* -algebra, let $\mathcal{P}(\mathcal{U})$ denote the sets of all projections in \mathcal{U} . If $P : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ is a holomorphic \mathcal{U} -valued map, then P is called an extended holomorphic curve on $\mathcal{P}(\mathcal{U})$. In this note, we focus on discussing curvature formulae of the extended holomorphic curves. By using the curvature formulae, we give a necessary and sufficient condition for some extended holomorphic curves on C^* -algebras to be unitary equivalent and also give a similarity theorem involving curvature and its partial derivatives for Cowen-Douglas operators.

1. INTRODUCTION

In this note, we will give the curvature formulae of extended holomorphic curves in Grassmann manifolds in a C^* -algebraic setting and discuss its application in Cowen-Douglas theory.

In Cowen-Douglas theory, holomorphic curve in Grassmann manifold is a basic and important concept. Let \mathcal{H} be a complex separable Hilbert space and $Gr(n, \mathcal{H})$ denote n -dimensional Grassmann manifold, the set of all n -dimensional subspaces of \mathcal{H} . A map $p : \Omega \rightarrow Gr(n, \mathcal{H})$ is called as a holomorphic curve, if there exist n holomorphic \mathcal{H} -valued functions e_1, e_2, \dots, e_n on Ω such that $p(\lambda) = \overline{\text{span}}\{e_1(\lambda), \dots, e_n(\lambda)\}$ for each $\lambda \in \Omega$, where symbol “ $\overline{\text{span}}$ ” denotes the closure of linear span (cf. [6]). And the concept of the extended holomorphic curve was first introduced by M. Martin and N. Salinas in [16]. It can be regarded as a generalization of classical holomorphic curve on Grassmann manifold. Let \mathcal{U} be a unital C^* -algebra, then $p \in \mathcal{U}$ is called a projection in \mathcal{U} whenever $p^2 = p = p^*$, and $\mathcal{P}(\mathcal{U})$ denote the set of all projections in \mathcal{U} which is called Grassmann manifold of \mathcal{U} . Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If $P : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ is a holomorphic \mathcal{U} -valued map, then it is called an extended holomorphic curve on $\mathcal{P}(\mathcal{U})$ (in order to discriminate ordinary holomorphic curve).

This class of holomorphic curves in a C^* -algebraic setting has been studied by C. Apostol, M. Martin, N. Salinas and D. R. Wilkins in a number of articles[1,16,18,19,26,29]. In 1981, a C^* -algebra approach to Cowen-Douglas theory was given by C. Apostol and M. Martin (cf. [1]). And M. Martin and N. Salinas did a series work of holomorphic curves on extended flag manifolds and extended Grassmann manifolds (cf. [16,18,19,26]). So this kind of researches on extended holomorphic curves can be regarded as one of generalization of Cowen-Douglas theory on C^* -algebras.

In the paper [6], M. J. Cowen and R. G. Douglas introduced a class of operators related to complex geometry now referred to as Cowen-Douglas operators [cf. Example 2.2]. There exists a natural connection between holomorphic curves and this class of operators. For \mathcal{H} a complex and separable Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the set of bounded linear operators on \mathcal{H} . Let Ω be

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a open connected subset of complex plane \mathbb{C} . A class of Cowen-Douglas operator with index one: $B_1(\Omega)$ is defined as follows[6]:

$$B_n(\Omega) =: \{T \in \mathcal{L}(\mathcal{H}) : \begin{aligned} &(i) \ \Omega \subset \sigma(T) =: \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}, \\ &(ii) \ \bigvee_{\lambda \in \Omega} \text{Ker}(T - \lambda) = \mathcal{H}, \\ &(iii) \ \text{Ran}(T - \lambda) = \mathcal{H}, \\ &(iv) \ \dim \text{Ker}(T - \lambda) = n, \forall \lambda \in \Omega. \end{aligned}\}$$

For any operator $T \in B_n(\Omega)$, it is shown that we can find a holomorphic family of eigenvectors $\{e_i(\lambda), \lambda \in \Omega\}_{i=1}^n$ such that $Te_i(\lambda) = \lambda e_i(\lambda), \forall \lambda \in \Omega$. A holomorphic curve with n dimension is a map from \mathcal{H} to Grassmann manifold $Gr(n, \mathcal{H})$ defined as $F(\lambda) =: \bigvee \{e_i(\lambda), i = 1, 2, \dots, n\}$ for $\lambda \in \Omega$.

M. J. Cowen and R. G. Douglas obtained a unitary equivalence classification of holomorphic curves in [6]. They proved that a kind of curvature function is a unitary invariant of the holomorphic curves and Cowen-Douglas operators by means of complex hermitian geometry techniques.

For any Cowen-Douglas operator T , there exists a Hermitian holomorphic bundle E_T with the fiber $F(\lambda), \lambda \in \Omega$. We call two linear bounded operators T and S are unitarily equivalent if and only if there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $T = USU^*$ (denoted by $T \sim_u S$). For two holomorphic curves F and G defined on Ω , if there exists a unitary operator $U \in \mathcal{L}(\mathcal{H})$ such that $F(\lambda) = UG(\lambda), \forall \lambda \in \Omega$, then we call them are unitarily equivalent (denoted by $F \sim_u G$).

In [6], it is shown that unitary equivalence of operator T can be deduced to the same problem of holomorphic curve F associate to it. Following M. I. Cowen and R. G. Douglas [6], a curvature function for $T \in B_n(\Omega)$ can be defined as:

$$K_T(\lambda) = -\frac{\partial}{\partial \bar{\lambda}}(h^{-1} \frac{\partial h}{\partial \lambda}), \text{ for all } \lambda \in \Omega,$$

where the metric

$$h(\lambda) = (\langle e_j(\lambda), e_i(\lambda) \rangle)_{n \times n}, \forall \lambda \in \Omega,$$

and $\{e_1(\lambda), e_2(\lambda), \dots, e_n(\lambda)\}$ are the frames of E_T .

Let E_T be a Hermitian holomorphic bundle induced by a Cowen-Douglas operator T , and K_T be a curvature of T . Then covariant partial derivatives of curvature K_T are defined as the following:

$$\begin{aligned} (1) \quad &K_{T, \bar{z}} = \frac{\partial}{\partial \bar{\lambda}}(K_T); \\ (2) \quad &K_{T, z} = \frac{\partial}{\partial \lambda}(K_T) + [h^{-1} \frac{\partial h}{\partial \lambda}, K_T]. \end{aligned}$$

And a remarkable result is also proved in [6]: For $T, S \in B_1(\Omega)$, $T \sim_u S$ if and only if $K_T = K_S$ on Ω .

And let $T_1, T_2 \in B_n(\Omega)$. Then $T_1 \sim_u T_2$ if and only if there exists an isometry $V : E_{T_1} \rightarrow E_{T_2}$ such that

$$VK_{T_1, z^i \bar{z}^j} = K_{T_2, z^i \bar{z}^j} V, i, j = 0, 1, \dots, n-1.$$

Subsequently, the curvature function turns into an important object of the research of Cowen-Douglas operators. R. G. Douglas, G. Misra, K. Guo, D. N. Clark, M. Uchiyama, H. Kwon, S. Treil, L. Chen, S. S. Roy and A. Korányi and many other mathematicians did a lot of work around the curvature ([2][3][4][5][6][7][14][23][26]). On the other hand, by using K_0 -group, C. Jiang, X. Guo and K. Ji concerned the problems of similarity classification of Cowen-Douglas operators and some holomorphic curves[8][9][10][11].

Same to the researches of Cowen-Douglas operators, we also start from the unitarily equivalence of this kind of holomorphic curves. Let $P, Q : \Omega \rightarrow P(\mathcal{U})$ be two extended holomorphic

curves. We say that P and Q are unitary equivalent (denoted by $P \stackrel{u}{\sim} Q$) if there exists a unitary $U \in \mathcal{U}$ such that $P(\lambda) = UQ(\lambda)U^*, \forall \lambda \in \Omega$ (cf. [16]).

In [16], M. Martin and N. Salinas give the unitarily invariants of extended holomorphic curve P by considering the partial derivatives $\partial^I P \bar{\partial}^J P$, $I, J \in \mathbb{N}$. As we mentioned above, the important and interesting part of the researches in holomorphic curves and Cowen-Douglas operators is the intrinsic connection with complex geometry. One can decide the unitarily equivalence of two operators by calculating their curvatures. From this view point, we also need to search the geometry unitarily invariants of extended holomorphic curve. So a natural question is the following :

Question 1 *What is the curvature for the extended holomorphic curves? And is it also the unitarily invariants of extended holomorphic curves?*

To answer this question, we want to characterize the curvature and it's covariant partial derivatives's formulaes and unitary equivalence problem of extended holomorphic curves with these geometry concepts.

On the other hand, people also consider the similarity classification of holomorphic curves and Cowen-Douglas operators. As we all known, curvature is not the similarity invariant (cf [3],[4]). In [11], we give a similarity classification of holomorphic curves involving the K_0 group of the holomorphic curve's commutant algebra. But we still have no any geometry invariants for the similarity of holomorphic curves and Cowen-Douglas operators.

In 2009, R. G. Douglas asked the following question:

Question 2 *Can one give conditions involving the curvatures which imply that two quasi-free Hilbert modules of multiplicity one are similar?*

In [14], by considering the Hilbert-Schmidt norm of the partial derivative of analytic projection (or the trace of the curvature of corresponding operator), H.Kwon and S.Treil characterize contractions with certain property that are similar to the backward shift in the Hardy space. This analytic projection is also a kind of extended holomorphic curve. And this result was also generalized to the weighted Bergman shift case by R. G. Douglas, H. Kwon and S. Treil (cf [7]). As an application, we find the relationship of the algebra invariant (K_0 -group) and geometry invariant (curvature) for Cowen-Douglas operators by using this curvature formulaes of extended holomorphic curves. And we also describe the trace of derivatives of curvatures for Cowen-Douglas operators in the form of extended holomorphic curves.

The paper is organized as follows. In §1 some notations and known results will be introduced. In §2, We define a special class of extended holomorphic curves analogous to Bott projection in C^* -algebras. We also give a curvature formulae for this kind of extended holomorphic curves and we also consider the unitarily classification of extended holomorphic curves by using this curvature. In §3, we give a similarity theorem involving curvature and it's partial derivatives for Cowen-Douglas operators. In §4, we introduce the relation between the curvature formulae and H.Kwon and S.Treil's work.

We will introduce some notations and results first, and all the notations are adopted from [6],[8] and [16].

To simplify the notation, we use the symbol " $\bar{\partial}^J \partial^I$ " denotes partial derivative " $\frac{\partial^{J+I}}{\partial^J \bar{\lambda} \partial^I \lambda}$ ", where I, J are non-negative integers. And for any I and J ,

- (1) symbol $\bar{\partial}^J$ stands for $\bar{\partial}^J \partial^0$ and ∂^I stands for $\bar{\partial}^0 \partial^I$,
- (2) symbol ∂ stands for ∂^1 , and $\bar{\partial}$ stands for $\bar{\partial}^1$,
- (3) $\bar{\partial}^J \partial^I P = P$, when $J = I = 0$.

Firstly, we need a criterion for determining the holomorphic map from Ω to $\mathcal{P}(\mathcal{U})$.

1.1[16] Let \mathcal{U} be a unital C^* -algebra. Let $P : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ be a \mathcal{U} -valued infinitely differentiable map. Then P is called holomorphic if and only if

$$\bar{\partial}P(\lambda) = P(\lambda)\bar{\partial}P(\lambda), \forall \lambda \in \Omega. \quad (1.1.1)$$

Since $P(\lambda)$ is a projection, for any $\lambda \in \Omega$, we can get that

$$\bar{\partial}P(\lambda) = [\bar{\partial}P(\lambda)]P(\lambda) + P(\lambda)[\bar{\partial}P(\lambda)].$$

So (1.1.1) is equivalent to say that

$$[\bar{\partial}P(\lambda)]P(\lambda) = 0 \iff \partial P(\lambda) = [\partial P(\lambda)]P(\lambda) \iff P(\lambda)\partial P(\lambda) = 0, \forall \lambda \in \Omega.$$

By a direct computation, we also have

$$\bar{\partial}\partial^J P = \partial^J P\bar{\partial}P - \bar{\partial}P\partial^J P - \sum_{k=1}^{J-1} C_J^k (\partial^{J-k} P\bar{\partial}P\partial^k P), \forall J \in \mathbb{N}. \quad (1.1.2)$$

$$\bar{\partial}^I \partial P = \partial P\bar{\partial}^I P - \bar{\partial}^I P\partial P - \sum_{k=1}^{I-1} C_I^k (\bar{\partial}^{I-k} P\partial P\bar{\partial}^k P), \forall I \in \mathbb{N}. \quad (1.1.3)$$

and

$$\bar{\partial}^J P P = P\partial^J P = 0, \forall I, J \in \mathbb{N}. \quad (1.1.4)$$

For the general case, every derivative $\bar{\partial}^J \partial^I P$, $I, J \in \mathbb{N}$ may be expressed as a sum of monomials of the form (See more details in [16])

$$\pm [\bar{\partial}^{I_1} P][\partial^{J_1} P] \dots [\bar{\partial}^{I_k} P][\partial^{J_k} P]$$

and

$$\pm [\partial^{J_1} P][\bar{\partial}^{I_1} P] \dots [\partial^{J_k} P][\bar{\partial}^{I_k} P]. \quad (1.1.5)$$

Example 1.2 A class of Cowen-Douglas operator with index n : $B_n(\Omega)$ is defined as follows[2]:

$$\begin{aligned} B_n(\Omega) =: \{T \in \mathcal{L}(\mathcal{H}) : & \quad (i) \ \Omega \subset \sigma(T) =: \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}, \\ & \quad (ii) \ \bigvee_{\lambda \in \Omega} \text{Ker}(T - \lambda) = \mathcal{H}, \\ & \quad (iii) \ \text{Ran}(T - \lambda) = \mathcal{H}, \\ & \quad (iv) \ \dim \text{Ker}(T - \lambda) = n, \forall \lambda \in \Omega.\} \end{aligned}$$

Let $T \in \mathcal{L}(\mathcal{H})$ be a Cowen-Douglas operator. For any $\lambda \in \Omega$, if $P(\lambda)$ is the projection from \mathcal{H} to $\text{Ker}(T - \lambda)$, then $P : \Omega \rightarrow \mathcal{P}(\mathcal{L}(\mathcal{H}))$ is an extended holomorphic curve.

1.3 Let \mathcal{U} be a unital C^* -algebra, and $P : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ be an extended holomorphic curve. For each $\lambda \in \Omega$ and every $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$, set

$$\mathcal{B}_\lambda^\alpha = \{\bar{\partial}^J P(\lambda)\partial^I P(\lambda) : I, J \in \mathbb{Z}_+, I, J \leq \alpha\}.$$

Let $\mathcal{U}_\lambda^\alpha$ be the closure of $*$ -subalgebra of \mathcal{U} generated by $\mathcal{B}_\lambda^\alpha$ with the following property:

$$\mathcal{U}_\lambda^0 \subseteq \mathcal{U}_\lambda^1 \subseteq \dots \subseteq \mathcal{U}_\lambda^\infty.$$

By using notations mentioned above, M. Martin and N. Salinas defined a substitute in C^* -algebra for Cowen-Douglas class $B_n(\Omega)$:

Definition 1.4[16] Let $k \geq 1$ be an integer. If the following conditions are satisfied, then extended holomorphic curve $P : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ is said to be in the class $\mathcal{A}_k(\Omega, \mathcal{U})$:

- (1) For each $\lambda \in \Omega$, $\mathcal{U}_\lambda^\infty$ is a finite-dimensional C^* -algebra.
- (2) If k_λ denotes the cardinal of any maximal collection of mutually orthogonal minimal projections in $\mathcal{U}_\lambda^\infty$, then

$$k_\lambda \leq k.$$

- (3) If $a \in \mathcal{U}$ and $aP(\lambda) = 0$ for every $\lambda \in \Omega$, then $a = 0$.

Definition 1.5[16] Let $\lambda \in \Omega$ and $\alpha \in \mathbb{Z}_+$ be a fixed integer. We say that P and Q **have order of contact α at λ** if there exists a unitary ν such that

$$\nu \bar{\partial}^J P(\lambda) \partial^I P(\lambda) \nu^* = \bar{\partial}^J Q(\lambda) \partial^I Q(\lambda), \quad \forall 0 \leq I, J \leq \alpha, \quad (1.2)$$

We say $\mathfrak{G} \subset \mathcal{U}$ is a **separating subset** of \mathcal{U} , if $\{a \in \mathcal{U} : as = 0, s \in \mathfrak{G}\} = \{0\}$. Assume $\mathfrak{G}, \mathfrak{T}$ are two separating subsets of \mathcal{U} , $\theta : \mathfrak{G} \rightarrow \mathfrak{T}$ is a given bijection. We say θ is **inner (or semi-inner)**, if there exists a unitary $u \in \mathcal{U}$ (or a unitary $\nu \in \mathcal{U}$) such that

$$usu^* = \theta(s), \quad \forall s \in \mathfrak{G}, \quad (\text{or } \nu t^* s \nu^* = \theta(t) \theta(s), \quad \forall s, t \in \mathfrak{G}).$$

\mathcal{U} is said to be inner if each semi-inner bijection between two separating subsets of \mathcal{U} is inner.

M.Martin and N.Salinas proved the following related rigidity theorem for $\mathcal{A}_k(\Omega, \mathcal{U})$ class on C^* -algebra.

Lemma 1.6[Theorem 4.5, 16] *Suppose that extended holomorphic curves $P, Q : \Omega \rightarrow \mathcal{P}(\mathcal{U})$ belong to the class $\mathcal{A}_k(\Omega, \mathcal{U})$. If \mathcal{U} is an inner C^* -algebra, then the following two statements are equivalent:*

- (1) P and Q are unitarily equivalent;
- (2) P and Q have order of contact α at each $\lambda \in \Omega$.

2. A GENERALIZATION OF $B_n(\Omega)$ ON C^* -ALGEBRAS

Let B be a unital C^* -algebra. A Hilbert B -module $l^2(\mathbb{N}, B)$ is defined as

$$l^2(\mathbb{N}, B) =: \{(a_i)_{i \in \mathbb{N}} : a_i \in B, \forall i \in \mathbb{N}, \text{ and } \sum_{i \in \mathbb{N}} \|a_i\|^2 < \infty\}.$$

We denote the set of all the linear bounded operators on $l^2(\mathbb{N}, B)$ by $\mathcal{L}(l^2(\mathbb{N}, B))$. Then $\mathcal{L}(l^2(\mathbb{N}, B))$ is a C^* -algebra.

Let

$$\alpha_i^* = (a_1^{j^*}, a_2^{j^*}, \dots, a_i^{j^*}, \dots) \in l^2(\mathbb{N}, B), i = 1, 2, \dots, n$$

and

$$\alpha_i = (a_1^j, a_2^j, \dots, a_i^j, \dots)^T, i = 1, 2, \dots, n$$

be the conjugate transpose of α_i^* .

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*)^T$ and the symbol “ \cdot ” denotes the multiplication of matrix. We will first introduce the following two notations:

(1)

$$\begin{aligned} \alpha \cdot \alpha^* : &= (\alpha_1, \alpha_2, \dots, \alpha_n) \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1^1 \alpha_1^1 \dots \alpha_n^1 \\ \alpha_1^2 \alpha_2^2 \dots \alpha_n^2 \\ \dots \dots \dots \\ \alpha_1^k \alpha_2^k \dots \alpha_n^k \\ \dots \dots \dots \end{pmatrix}_{\infty \times n} \begin{pmatrix} \alpha_1^{1*} \alpha_1^{*2} \dots \alpha_1^{l*} \dots \\ \alpha_2^{1*} \alpha_2^{*2} \dots \alpha_2^{l*} \dots \\ \dots \dots \dots \\ \alpha_n^{1*} \alpha_n^{*2} \dots \alpha_n^{l*} \dots \end{pmatrix}_{n \times \infty} \end{aligned}$$

which can be seen as an element in $\mathcal{L}(l^2(\mathbb{N}, B))$;

(2)

$$\begin{aligned} \alpha^* \cdot \alpha : &= \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} (\alpha_1, \alpha_2, \dots, \alpha_n) \\ &= \begin{pmatrix} \alpha_1^{1*} \alpha_1^{*2} \cdots \alpha_1^{l*} \cdots \\ \alpha_2^{1*} \alpha_2^{*2} \cdots \alpha_2^{l*} \cdots \\ \vdots \\ \alpha_n^{1*} \alpha_n^{*2} \cdots \alpha_n^{l*} \cdots \end{pmatrix}_{n \times \infty} \begin{pmatrix} \alpha_1^1 \alpha_2^1 \cdots \alpha_n^1 \\ \alpha_1^2 \alpha_2^2 \cdots \alpha_n^2 \\ \vdots \\ \alpha_1^k \alpha_2^k \cdots \alpha_n^k \\ \vdots \end{pmatrix}_{\infty \times n} \\ &= \begin{pmatrix} \sum_{i=1}^{\infty} \alpha_1^{i*} \alpha_1^i & \sum_{i=1}^{\infty} \alpha_1^{i*} \alpha_2^i & \cdots & \sum_{i=1}^{\infty} \alpha_1^{i*} \alpha_n^i \\ \sum_{i=1}^{\infty} \alpha_2^{i*} \alpha_1^i & \sum_{i=1}^{\infty} \alpha_2^{i*} \alpha_2^i & \cdots & \sum_{i=1}^{\infty} \alpha_2^{i*} \alpha_n^i \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{\infty} \alpha_n^{i*} \alpha_1^i & \sum_{i=1}^{\infty} \alpha_n^{i*} \alpha_2^i & \cdots & \sum_{i=1}^{\infty} \alpha_n^{i*} \alpha_n^i \end{pmatrix}_{n \times n} \end{aligned}$$

Definition 2.1. Let Ω be a connected open subset of \mathbb{C} and B be a unital C^* -algebra. For $\mathcal{U} = \mathcal{L}(l^2(\mathbb{N}, B))$, let $\mathcal{P}_n(\Omega, \mathcal{U})$ denotes the extended holomorphic curve P which satisfies:

$$P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda), \forall \lambda \in \Omega, \text{ where}$$

$$\alpha_i^* = (a_1^{j^*}, a_2^{j^*}, \dots, a_i^{j^*}, \dots) \in l^2(\mathbb{N}, B), i = 1, 2, \dots, n$$

and

$$\alpha_i = (a_1^j, a_2^j, \dots, a_i^j, \dots)^T, i = 1, 2, \dots, n$$

and

$\alpha_i^j : \Omega \rightarrow l^2(\mathbb{N}, B)$ is a holomorphic function and a_i^{j*} is the conjugate transpose of α .

Example 2.2. Let $E(\lambda), \lambda \in \mathbb{D}$ be an analytic family of subspaces of Hilbert space \mathcal{H} (or holomorphic curve). And let $P(\lambda)$ be the orthogonal projection onto $E(\lambda)$. Then $P : \mathbb{D} \rightarrow P(\mathcal{L}(\mathcal{H}))$ is an extended holomorphic curve (cf [14] [16]). As we all known, the subspace $E(\lambda)$ is equal to the range of $F(\lambda)$ where F is a left invertible analytic operator-valued function. And

$$P = F(F^*F)^{-1}F^*.$$

In Definition 2.1, when we assume $\mathcal{U} = \mathcal{L}(\mathcal{H})$, we can see that $\{\alpha_i(\lambda)\}_{i=1}^n$ are the frames of $E(\lambda) = \text{Ran}F(\lambda)$ for any $\lambda \in \mathbb{D}$.

Example 2.3. [8] For the finite dimension case, let \mathcal{U} be $M_2(\mathbb{C})$ and $\Omega \subseteq \mathbb{C}$ be a connected open set, and let $P : \Omega \rightarrow M_2(\mathbb{C})$ defined by

$$P(\lambda) = \frac{1}{1+|\lambda|^2} \begin{pmatrix} 1 & \bar{\lambda} \\ \lambda & |\lambda|^2 \end{pmatrix}, \forall \lambda \in \Omega.$$

Then P is called Bott projection in algebra K-theory. When we assume that $\alpha(\lambda) := (1, \lambda)^T \in \mathbb{C}^2$ and $\alpha^*(\lambda) := (1, \bar{\lambda})$, then we have

$$P(\lambda) = (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \alpha(\lambda) \cdot \alpha^*(\lambda), \forall \lambda \in \Omega.$$

And P is an extended holomorphic curve on Ω .

Definition 2.4. Let $P \in \mathcal{P}_n(\Omega, \mathcal{U})$. Considering $l^2(\mathbb{N}, B)$ is a Hilbert C^* -module, denote the metric

$$h(\lambda) = \langle \alpha(\lambda), \alpha(\lambda) \rangle = \alpha^*(\lambda) \cdot \alpha(\lambda).$$

An curvature function of P is defined as

$$K_P = -\frac{\partial}{\partial \bar{\lambda}}(h^{-1} \frac{\partial h}{\partial \lambda}), \text{ for all } \lambda \in \Omega.$$

And the partial derivatives of curvature are defined as the following:

- (1) $K_{P, \bar{\lambda}} = \frac{\partial}{\partial \bar{\lambda}}(K_P)$;
- (2) $K_{P, \lambda} = \frac{\partial}{\partial \lambda}(K_P) + [h^{-1} \frac{\partial}{\partial \lambda} h, K_P]$, for all $\lambda \in \Omega$.

By the definition above, we can get the partial derivatives of curvature: $K_{P, \lambda^i \bar{\lambda}^j}$, $i, j \in \mathbb{N} \cup \{0\}$ by using the inductive formulae above. And this curvature and the partial derivatives of curvature are same to the curvature of Cowen-Douglas operator in form.

Proposition 2.5. *Let B be a unital C^* -algebra and $\mathcal{U} = \mathcal{L}(l^2(\mathbb{N}, B))$. Let $\alpha : \Omega \rightarrow l^2(\mathbb{N}, B)$ be a holomorphic function. Then the C^∞ map $P := \alpha \cdot (\alpha^* \cdot \alpha)^{-1} \cdot \alpha^*$ defined in definition 2.2 is an extended holomorphic curve.*

Proof. Firstly, note that

$$\begin{aligned} h &= \alpha^* \cdot \alpha \\ &= \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} (\alpha_1, \alpha_2, \dots, \alpha_n). \end{aligned}$$

Then we have

$$\begin{aligned} P^2 &= (\alpha \cdot (\alpha^* \cdot \alpha)^{-1} \cdot \alpha^*) (\alpha \cdot (\alpha^* \cdot \alpha)^{-1} \cdot \alpha^*) \\ &= (\alpha_1, \alpha_2, \dots, \alpha_n) h^{-1} \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} (\alpha_1, \alpha_2, \dots, \alpha_n) h^{-1} \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} \\ &= (\alpha_1, \alpha_2, \dots, \alpha_n) h^{-1} h h^{-1} \begin{pmatrix} \alpha_1^* \\ \alpha_2^* \\ \vdots \\ \alpha_n^* \end{pmatrix} \\ &= \alpha \cdot (\alpha^* \cdot \alpha)^{-1} \cdot \alpha^* \\ &= P \end{aligned}$$

and $P^* = P$, then $P(\lambda)$ is orthogonal projection for any $\lambda \in \Omega$.

By Definition 1.2, we only need to prove that P satisfies the formulae (1.1). Note that $\bar{\partial}(\alpha) = 0$, we have that

$$\begin{aligned} \bar{\partial} P P &= \bar{\partial}(\alpha(\alpha^* \alpha)^{-1} \alpha^*) (\alpha(\alpha^* \alpha)^{-1} \alpha^*) \\ &= \bar{\partial}(\alpha h^{-1} \alpha^*) (\alpha h^{-1} \alpha^*) \\ &= (\alpha \bar{\partial}(h^{-1}) \alpha^* + \alpha h^{-1} \bar{\partial} \alpha^*) (\alpha h^{-1} \alpha^*) \\ &= (\alpha \bar{\partial}(h^{-1}) \alpha^* \alpha h^{-1} \alpha^* + \alpha h^{-1} \bar{\partial} \alpha^* \alpha h^{-1} \alpha^*) \\ &= 0. \end{aligned}$$

This finishes the proof of lemma 2.1. □

Lemma 2.6. *Let $P \in \mathcal{P}_n(\Omega, \mathcal{U}) \cap \mathcal{A}_n(\Omega, \mathcal{U})$. And there exist holomorphic functions $\alpha : \Omega \rightarrow l^2(\mathbb{N}, B)$ such that*

$$P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda).$$

Then there exist fixed $F_{i,j}(P)$, $i, j = 0, 1, \dots, n$ which are linear combinations of $\bar{\partial}^{J_1} P \partial^{I_1} P \dots \bar{\partial}^{J_k} P \partial^{I_k} P$ such that the following conclusion hold:

$$F_{i,j}(P)(\lambda) = \alpha(\lambda)(-K_{P,z^i,\bar{z}^j})h_1^{-1}\alpha^*(\lambda), \forall \lambda \in \Omega.$$

Proof. Let $P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda) = \alpha(\lambda) \cdot h^{-1}(\lambda) \cdot \alpha^*(\lambda)$, $\forall \lambda \in \Omega$. And

$$\alpha(\lambda) = (\alpha_1, \alpha_2, \dots, \alpha_n), \alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*)^T$$

where

$$\alpha_i^* = (a_1^{j^*}, a_2^{j^*}, \dots, a_i^{j^*}, \dots) \in l^2(\mathbb{N}, B), i = 1, 2, \dots, n$$

$$\alpha_i = (a_1^j, a_2^j, \dots, a_i^j, \dots)^T \in l^2(\mathbb{N}, B), i = 1, 2, \dots, n,$$

and

$$h(\lambda) = \alpha^*(\lambda) \cdot \alpha(\lambda).$$

Then we have the following claim:

Claim 1

$$\partial^I P = (\partial^I \alpha h^{-1} + C_I^1 \partial^{I-1} \alpha \partial h^{-1} + \dots + C_I^k \partial^{I-k} \alpha \partial^k h^{-1} + \dots + \alpha \partial^I h^{-1}) \alpha^*, \forall I \in \mathbb{N}, \quad (2.6.1)$$

$$\bar{\partial}^J P = \alpha(\bar{\partial}^J h^{-1} \alpha^* + C_J^1 \bar{\partial}^{J-1} h^{-1} \bar{\partial} \alpha^* + \dots + C_J^k \bar{\partial}^{J-k} h^{-1} \bar{\partial}^k \alpha^* + \dots + h^{-1} \bar{\partial}^J \alpha^*), \forall J \in \mathbb{N}. \quad (2.6.2)$$

Since $(\partial^I P)^* = \bar{\partial}^I P$, $\forall I \in \mathbb{N}$, then we only need to prove the formulae (2.6.2). When $I = 1$, note that $\bar{\partial} \alpha = 0$, we have that

$$\begin{aligned} \bar{\partial} P &= \bar{\partial}(\alpha \cdot h^{-1} \cdot \alpha^*) \\ &= \alpha \bar{\partial} h^{-1} \alpha^* + \alpha h^{-1} \bar{\partial} \alpha^*. \end{aligned}$$

By induction proof, suppose that

$$\bar{\partial}^{J-1} P = \alpha(\bar{\partial}^{J-1} h^{-1} \alpha^* + C_{J-1}^1 \bar{\partial}^{J-2} h^{-1} \bar{\partial} \alpha^* + \dots + C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^k \alpha^* + \dots + h^{-1} \bar{\partial}^{J-1} \alpha^*).$$

Then we have

$$\begin{aligned} \bar{\partial}(\bar{\partial}^{J-1} P) &= \bar{\partial}(\alpha(\bar{\partial}^{J-1} h^{-1} \alpha^* + \dots + C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^k \alpha^* + \dots + h^{-1} \bar{\partial}^{J-1} \alpha^*)) \\ &= \alpha(\bar{\partial}^J h^{-1} \alpha^* + \bar{\partial}^{J-1} h^{-1} \bar{\partial} \alpha^* + \dots + \bar{\partial}(C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^k \alpha^*) + \dots \\ &\quad + \bar{\partial} h^{-1} \bar{\partial}^{J-1} \alpha^* + h^{-1} \bar{\partial}^J \alpha^*). \end{aligned}$$

Note that

$$\begin{aligned} \bar{\partial}(C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^k \alpha^*) &= C_{J-1}^k \bar{\partial}^{J-k} h^{-1} \bar{\partial}^k \alpha^* + C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^{k+1} \alpha^*, \\ \bar{\partial}(C_{J-1}^{k+1} \bar{\partial}^{J-k-2} h^{-1} \bar{\partial}^{k+1} \alpha^*) &= C_{J-1}^{k+1} \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^{k+1} \alpha^* + C_{J-1}^{k+1} \bar{\partial}^{J-k-2} h^{-1} \bar{\partial}^{k+2} \alpha^* \end{aligned}$$

and

$$C_{J-1}^k \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^{k+1} \alpha^* + C_{J-1}^{k+1} \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^{k+1} \alpha^* = C_J^{k+1} \bar{\partial}^{J-k-1} h^{-1} \bar{\partial}^{k+1} \alpha^*$$

Then we have

$$\bar{\partial}^J P = \alpha(\bar{\partial}^J h^{-1} \alpha^* + C_J^1 \bar{\partial}^{J-1} h^{-1} \bar{\partial} \alpha^* + \dots + C_J^k \bar{\partial}^{J-k} h^{-1} \bar{\partial}^k \alpha^* + \dots + h^{-1} \bar{\partial}^J \alpha^*), \forall J \in \mathbb{N}.$$

So we finish the proof of Claim 1.

Claim 2

$$\bar{\partial} P \partial P = \alpha(-K_P) h^{-1} \alpha^*; \quad (2.6.3)$$

$$\bar{\partial} P \partial^2 P = \alpha(-(K_P)_z) h^{-1} \alpha^*; \quad (2.6.4)$$

$$\bar{\partial}^2 P \partial P = \alpha(-(K_P)_{\bar{z}}) h^{-1} \alpha^*; \quad (2.6.5)$$

$$\bar{\partial}^2 P \partial^2 P - 2(\bar{\partial} P \partial P)^2 = \alpha(-(K_P)_{z\bar{z}}) h^{-1} \alpha^*; \quad (2.6.6)$$

Note that $K_P = -(\bar{\partial}h^{-1}\partial h + h^{-1}\bar{\partial}\partial h)$, and $\theta_P = h^{-1}\partial h$. Then we have that

$$\begin{aligned}
\bar{\partial}P\partial^2P &= (\alpha\bar{\partial}h^{-1}\alpha^* + \alpha h^{-1}\bar{\partial}\alpha^*)(\partial^2\alpha h^{-1}\alpha^* + 2\partial^1\alpha\partial h^{-1}\alpha^* + \alpha\partial^2h^{-1}\alpha^*) \\
&= \alpha(\bar{\partial}h^{-1}\partial^2h + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h + h^{-1}\partial\bar{\partial}^2h)h^{-1}\alpha^* \\
&= \alpha(\bar{\partial}h^{-1}\partial^2h + h^{-1}\partial\bar{\partial}^2h + \bar{\partial}\partial h^{-1}\partial h + \partial h^{-1}\bar{\partial}\partial h \\
&\quad + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h - \bar{\partial}\partial h^{-1}\partial h - \partial h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h - \bar{\partial}\partial h^{-1}\partial h - \partial h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h + \bar{\partial}(h^{-1}\partial h h^{-1})\partial h - \partial h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h + \bar{\partial}h^{-1}\partial h h^{-1}\partial h + h^{-1}\partial\bar{\partial}h h^{-1}\partial h \\
&\quad + h^{-1}\partial h\bar{\partial}h^{-1}\partial h - h^{-1}\partial h h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^*
\end{aligned}$$

Note that $\partial h^{-1}h = -h^{-1}\partial h = -\theta_P$, we have

$$\begin{aligned}
\bar{\partial}P\partial^2P &= \alpha(-\partial K_P + 2\bar{\partial}h^{-1}\partial h\partial h^{-1}h + 2h^{-1}\partial\bar{\partial}h\partial h^{-1}h + \bar{\partial}h^{-1}\partial h h^{-1}\partial h + h^{-1}\partial\bar{\partial}h h^{-1}\partial h \\
&\quad + h^{-1}\partial h\bar{\partial}h^{-1}\partial h + h^{-1}\partial h h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P + \bar{\partial}h^{-1}\partial h\partial h^{-1}h + h^{-1}\partial\bar{\partial}h\partial h^{-1}h + h^{-1}\partial h\bar{\partial}h^{-1}\partial h + h^{-1}\partial h h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P - \bar{\partial}h^{-1}\partial h h^{-1}\partial h - h^{-1}\partial\bar{\partial}h h^{-1}\partial h + h^{-1}\partial h\bar{\partial}h^{-1}\partial h + h^{-1}\partial h h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= \alpha(-\partial K_P - \bar{\partial}h^{-1}\partial h\theta_P - h^{-1}\partial\bar{\partial}h\theta_P + \theta_P\bar{\partial}h^{-1}\partial h + \theta_P h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= -\alpha(\partial K_P + \bar{\partial}h^{-1}\partial h\theta_P + h^{-1}\partial\bar{\partial}h\theta_P - \theta_P\bar{\partial}h^{-1}\partial h - \theta_P h^{-1}\bar{\partial}\partial h)h^{-1}\alpha^* \\
&= -\alpha(\partial K_P + (\bar{\partial}h^{-1}\partial h + h^{-1}\partial\bar{\partial}h)\theta_P - \theta_P(\bar{\partial}h^{-1}\partial h - h^{-1}\bar{\partial}\partial h))h^{-1}\alpha^* \\
&= -\alpha(\partial K_P + (\theta_P K_P - K_P\theta_P))h^{-1}\alpha^*
\end{aligned}$$

By Definition 2.4,

$$(K_P)_z = \partial K_P + [\theta_P, K_P].$$

Then it follows that

$$\bar{\partial}P\partial^2P = \alpha(-(K_P)_z)h^{-1}\alpha^*.$$

And

$$\begin{aligned}
\bar{\partial}^2P\partial P &= \alpha\bar{\partial}(\bar{\partial}h^{-1}\alpha^* + h^{-1}\bar{\partial}\alpha^*)(\alpha\partial h^{-1} + \alpha\partial h^{-1})\alpha^* \\
&= \alpha(\bar{\partial}^2h^{-1}\alpha^* + 2\bar{\partial}h^{-1}\bar{\partial}\alpha^* + h^{-1}\bar{\partial}^2\alpha^*)(\alpha\partial h^{-1} + \alpha\partial h^{-1})\alpha^* \\
&= \alpha(\bar{\partial}^2h^{-1}\partial h h^{-1} + \bar{\partial}^2h^{-1}h\partial h^{-1} + 2\bar{\partial}h^{-1}\partial\bar{\partial}h h^{-1} + 2\bar{\partial}h^{-1}\bar{\partial}h\partial h^{-1} \\
&\quad + h^{-1}\bar{\partial}^2\partial h + h^{-1}\bar{\partial}^2h\partial h^{-1})\alpha^* \\
&= \alpha(\bar{\partial}^2h^{-1}\partial h + \bar{\partial}^2h^{-1}h\partial h^{-1}h + 2\bar{\partial}h^{-1}\partial\bar{\partial}h + 2\bar{\partial}h^{-1}\bar{\partial}h\partial h^{-1}h \\
&\quad + h^{-1}\bar{\partial}^2\partial h + h^{-1}\bar{\partial}^2h\partial h^{-1}h)h^{-1}\alpha^* \\
&= \alpha(-\bar{\partial}K + (\bar{\partial}^2h^{-1}h + 2\bar{\partial}h^{-1}\bar{\partial}h + h^{-1}\bar{\partial}^2h)\partial h^{-1}h)h^{-1}\alpha^* \\
&= \alpha(-\bar{\partial}K + \bar{\partial}(\bar{\partial}h^{-1}h + h^{-1}\bar{\partial}h)\partial h^{-1}h)h^{-1}\alpha^* \\
&= \alpha(-\bar{\partial}K)h^{-1}\alpha^* \\
&= \alpha(-K_P)\bar{z}h^{-1}\alpha^*.
\end{aligned}$$

Then it follows that

$$\bar{\partial}^2P\partial P = \alpha(-(K_P)\bar{z})h^{-1}\alpha^*.$$

And we also have

$$\begin{aligned}
\bar{\partial}^2P\partial^2P &= \alpha(\bar{\partial}^2h^{-1}\alpha^* + 2\bar{\partial}h^{-1}\bar{\partial}\alpha^* + h^{-1}\bar{\partial}^2\alpha^*)(\partial^2\alpha h^{-1} + 2\partial^1\alpha\partial h^{-1} + \alpha\partial^2h^{-1})\alpha^* \\
&= \alpha(\bar{\partial}^2h^{-1}\partial^2h + 2\bar{\partial}^2h^{-1}\partial h\partial h^{-1}h + \bar{\partial}^2h^{-1}h\partial^2h^{-1}h + 2\bar{\partial}h^{-1}\bar{\partial}\partial^2h \\
&\quad + 4\bar{\partial}h^{-1}\partial\bar{\partial}h\partial h^{-1}h + 2\bar{\partial}h^{-1}\bar{\partial}h\partial^2h^{-1}h + h^{-1}\bar{\partial}^2\partial^2h \\
&\quad + 2h^{-1}\bar{\partial}^2\partial h\partial h^{-1}h + h^{-1}\bar{\partial}^2h\partial^2h^{-1}h)h^{-1}\alpha^* \\
&= \alpha(\bar{\partial}^2h^{-1}\partial^2h + 2\bar{\partial}^2h^{-1}\partial h\partial h^{-1}h + 2\bar{\partial}h^{-1}\bar{\partial}^2\partial h + 4\bar{\partial}h^{-1}\partial\bar{\partial}h\partial h^{-1}h \\
&\quad + h^{-1}\bar{\partial}^2\partial^2h + 2h^{-1}\bar{\partial}^2\partial h\partial h^{-1}h)h^{-1}\alpha^*.
\end{aligned}$$

(2.6.7)

By formula in Definition 2.4, we have

$$\begin{aligned} -(K_P)_z &= -(\partial K_P + [\theta_P, K_P]) \\ &= \bar{\partial} h^{-1} \partial^2 h + 2\bar{\partial} h^{-1} \partial h \partial h^{-1} h + 2h^{-1} \partial \bar{\partial} h \partial h^{-1} h + h^{-1} \partial^2 \bar{\partial} h. \end{aligned}$$

and

$$\begin{aligned} -(K_P)_{z\bar{z}} = -\bar{\partial}((K_P)_z) &= \bar{\partial}(\bar{\partial} h^{-1} \partial^2 h + 2\bar{\partial} h^{-1} \partial h \partial h^{-1} h + 2h^{-1} \partial \bar{\partial} h \partial h^{-1} h + h^{-1} \partial^2 \bar{\partial} h) \\ &= \bar{\partial}^2 h^{-1} \partial^2 h + \bar{\partial} h^{-1} \bar{\partial} \partial^2 h + 2\bar{\partial}^2 h^{-1} \partial h \partial h^{-1} h + 2\bar{\partial} h^{-1} \partial \bar{\partial} h \partial h^{-1} h \\ &\quad + 2\bar{\partial} h^{-1} \partial h \partial \bar{\partial} h^{-1} h + 2\bar{\partial} h^{-1} \partial h \partial h^{-1} \bar{\partial} h + 2\bar{\partial} h^{-1} \partial \bar{\partial} h \partial h^{-1} h \\ &\quad + 2\bar{\partial} h^{-1} \partial \bar{\partial}^2 h \partial h^{-1} h + 2h^{-1} \partial \bar{\partial} h \partial \bar{\partial} h^{-1} h + 2h^{-1} \partial \bar{\partial} h \partial h^{-1} \bar{\partial} h \\ &\quad + \bar{\partial} h^{-1} \partial^2 \bar{\partial} h + h^{-1} \partial^2 \bar{\partial}^2 h. \end{aligned} \tag{2.6.8}$$

Note that

$$\begin{aligned} -2K_P^2 &= 2(\bar{\partial}(h^{-1} \partial h))(\bar{\partial}(\partial h^{-1} h)) \\ &= 2(\bar{\partial} h^{-1} \partial h + h^{-1} \partial \bar{\partial} h)(\bar{\partial} \bar{\partial} h^{-1} h + \partial h^{-1} \bar{\partial} h) \\ &= 2(\bar{\partial} h^{-1} \partial h \partial \bar{\partial} h^{-1} h + h^{-1} \partial \bar{\partial} h \partial \bar{\partial} h^{-1} h + \bar{\partial} h^{-1} \partial h \partial h^{-1} \bar{\partial} h \\ &\quad + h^{-1} \partial \bar{\partial} h \partial h^{-1} \bar{\partial} h) \end{aligned} \tag{2.6.9}$$

By formulae (2.6.7), (2.6.8) and (2.6.9), we have that

$$\begin{aligned} \bar{\partial}^2 P \partial^2 P + 2\alpha(-K_P^2)h^{-1}\alpha^* &= \bar{\partial}^2 P \partial^2 P - 2\alpha(-K_P)h^{-1}\alpha^* \alpha(-K_P)h^{-1}\alpha^* \\ &= \bar{\partial}^2 P \partial^2 P - 2(\bar{\partial} P \partial P)^2 \\ &= -\alpha((K_P)_{z,\bar{z}})h^{-1}\alpha^* \end{aligned}$$

Then it follows that

$$\bar{\partial}^2 P \partial^2 P - 2(\bar{\partial} P \partial P)^2 = \alpha(-(K_P)_{z,\bar{z}})h^{-1}\alpha^*.$$

This finishes the proof of Claim 2.

Claim 3 There exist fixed $F_{i,j}(P)$, $i, j = 0, 1, \dots, n$ which are linear combinations of $\bar{\partial}^{J_1} q \partial^{I_1} q \dots \bar{\partial}^{J_k} q \partial^{I_k} q$ such that

$$F_{i,j}(P)(\lambda) = \alpha(\lambda)(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*(\lambda), \forall \lambda \in \Omega.$$

Note that Claim 3 holds for $n = 2$, by induction proof, we assume that Claim 3 holds for $n \leq k$ and we will prove it also holds for $n = k + 1$ in the following.

Recall that

$$P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda) = \alpha(\lambda) \cdot h^{-1}(\lambda) \cdot \alpha^*(\lambda), \forall \lambda \in \Omega.$$

Set

$$\begin{aligned} \partial P &= F_1 + F_2, F_1 = \partial \alpha h^{-1} \alpha^*, F_2 = \alpha \partial h^{-1} \alpha^*; \\ \bar{\partial} P &= G_1 + G_2, G_1 = \alpha \bar{\partial} h^{-1} \alpha^*, G_2 = \alpha h^{-1} \bar{\partial} \alpha^*. \end{aligned}$$

Now suppose that $i = k$, or $j = k$ and

$$F_{i,j}(P)(\lambda) = \alpha(\lambda)(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*(\lambda), \forall \lambda \in \Omega.$$

Then we have

$$\begin{aligned} \partial(F_{i,j}(P)) &= \partial(\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \partial \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* + \alpha(-\partial(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* + \alpha(-(K_P)_{z^i, \bar{z}^j})\partial h^{-1}\alpha^*. \end{aligned} \tag{2.6.10}$$

Note that

$$\begin{aligned} F_1(F_{i,j}(P)) &= \partial \alpha h^{-1} \alpha^* (\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \partial \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* \end{aligned} \tag{2.6.11}$$

and

$$\begin{aligned}(F_{i,j}(P))F_2 &= (\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*)\alpha\partial h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})\partial h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})(-\theta_P)h^{-1}\alpha^*\end{aligned}\tag{2.6.12}$$

and

$$\begin{aligned}(F_{i,j}(P))F_1 &= (\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*)\partial\alpha h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\partial h h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})\theta_P h^{-1}\alpha^*\end{aligned}\tag{2.6.13}$$

and

$$\begin{aligned}F_2(F_{i,j}(P)) &= \alpha\partial h^{-1}\alpha^*(\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \alpha\partial h^{-1}h(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* \\ &= \alpha(-\theta_P)(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*.\end{aligned}\tag{2.6.14}$$

By formulae (2.6.10), (2.6.11) and (2.6.12), we have

$$\partial(F_{i,j}(P)) = \alpha(-\partial(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* + F_1(F_{i,j}(P)) + (F_{i,j}(P))F_2.\tag{2.6.15}$$

By formulae (2.6.13) and (2.6.14), we have

$$\alpha[\theta_P, -(K_P)_{z^i, \bar{z}^j}]h^{-1}\alpha^* = -F_2(F_{i,j}(P)) - (F_{i,j}(P))F_1.\tag{2.6.16}$$

Thus, it follows that

$$\begin{aligned}\alpha(-(K_P)_{z^{i+1}, \bar{z}^j})h^{-1}\alpha^* &= \alpha((-\partial(K_P)_{z^i, \bar{z}^j}) + [\theta, -(K_P)_{z^i, \bar{z}^j}])h^{-1}\alpha^* \\ &= \partial(F_{i,j}(P)) - F_1(F_{i,j}(P)) - (F_{i,j}(P))F_2 - F_2(F_{i,j}(P)) - (F_{i,j}(P))F_1 \\ &= \partial(F_{i,j}(P)) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P.\end{aligned}$$

Then we get the following induction formulae :

$$F_{i+1,j}(P) = \partial(F_{i,j}(P)) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P, i, j = 0, 1, \dots.\tag{2.6.17}$$

On the other hand, we have

$$\begin{aligned}\bar{\partial}(F_{i,j}(P)) &= \bar{\partial}(\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \alpha(-\bar{\partial}(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^* + \alpha(-(K_P)_{z^i, \bar{z}^j})\bar{\partial}h^{-1}\alpha^* + \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\bar{\partial}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^{j+1}})h^{-1}\alpha^* + \alpha(-(K_P)_{z^i, \bar{z}^j})\bar{\partial}h^{-1}\alpha^* + \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\bar{\partial}\alpha^*\end{aligned}\tag{2.6.18}$$

Note that

$$\begin{aligned}G_1(F_{i,j}(P)) &= \alpha\bar{\partial}h^{-1}\alpha^*(\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \alpha\bar{\partial}h^{-1}h(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*\end{aligned}\tag{2.6.19}$$

and

$$\begin{aligned}(F_{i,j}(P))G_2 &= (\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*)\alpha h^{-1}\bar{\partial}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*\alpha h^{-1}\bar{\partial}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\bar{\partial}\alpha^*\end{aligned}\tag{2.6.20}$$

and

$$\begin{aligned}(F_{i,j}(P))G_1 &= (\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*)\alpha\bar{\partial}h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}h\bar{\partial}h^{-1}\alpha^* \\ &= \alpha(-(K_P)_{z^i, \bar{z}^j})\bar{\partial}h^{-1}\alpha^*\end{aligned}\tag{2.6.21}$$

and

$$\begin{aligned}G_2(F_{i,j}(P)) &= \alpha h^{-1}\bar{\partial}\alpha^*(\alpha(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*) \\ &= \alpha h^{-1}\bar{\partial}h(-(K_P)_{z^i, \bar{z}^j})h^{-1}\alpha^*\end{aligned}\tag{2.6.22}$$

By formulae (2.6.20) and (2.6.21), we have

$$\bar{\partial}(F_{i,j}(P)) = \alpha(-(K_P)_{z^i, \bar{z}^{j+1}})h^{-1}\alpha^* + (F_{i,j}(P))G_1 + (F_{i,j}(P))G_2$$

By formulaes (2.6.19) and (2.6.22), we have that

$$G_1(F_{i,j}(P)) + G_2(F_{i,j}(P)) = 0.$$

Then we have

$$\begin{aligned} \bar{\partial}(F_{i,j}(P)) &= \alpha(-(K_P)_{z^i, \bar{z}^{j+1}})h^{-1}\alpha^* + (F_{i,j}(P))G_1 + (F_{i,j}(P))G_2 \\ &= \alpha(-(K_P)_{z^i, \bar{z}^{j+1}})h^{-1}\alpha^* + (F_{i,j}(P))G_1 + (F_{i,j}(P))G_2 + G_1(F_{i,j}(P)) + G_2(F_{i,j}(P)) \\ &= \alpha(-(K_P)_{z^i, \bar{z}^{j+1}})h^{-1}\alpha^* + \bar{\partial}P(F_{i,j}(P)) + (F_{i,j}(P))\bar{\partial}P \end{aligned}$$

Then we get the following induction formulae :

$$F_{i,j+1}(P) = \bar{\partial}(F_{i,j}(P)) - \bar{\partial}P(F_{i,j}(P)) - (F_{i,j}(P))\bar{\partial}P, i, j = 0, 1, \dots \quad (2.6.23)$$

By the Claim 2 and the induction formulae (2.6.17) and (2.6.23) i.e.

$$\begin{aligned} F_{i+1,j}(P) &= \partial(F_{i,j}(P)) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P, \\ F_{i,j+1}(P) &= \bar{\partial}(F_{i,j}(P)) - \bar{\partial}P(F_{i,j}(P)) - (F_{i,j}(P))\bar{\partial}P, i, j = 0, 1, \dots, \end{aligned}$$

we can find $F_{i,j}(P)$, $i, j = 0, 1, \dots$ such that

$$F_{i,j}(P)(\lambda) = \alpha(\lambda)(-K_{P,z^i, \bar{z}^j})h_1^{-1}\alpha^*(\lambda), \forall \lambda \in \Omega.$$

Claim 4 Each $F_{i,j}(P)$ for arbitrary i, j , may be expressed by as a sum of monomials of the form

$$(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t}.$$

By Claim 2, we already know that Claim 4 holds for the case of $i, j \leq 2$. By the induction proof, we assume that the conclusion holds for the case of $i, j \leq k$. Then we only need to prove the conclusion also holds for the case of $i, j \leq k+1$.

With loss of generality, when $i = k+1$ or $j = k+1$, we assume that

$$F_{i,j}(P) = (\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t}.$$

By (1.1.4) and (1.1.5), we also have

$$\begin{aligned} F_{i,j}(P) &= (\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t} \\ &= (\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t} P \\ &= F_{i,j}(P)P. \end{aligned}$$

By (1.1.4), it follows that

$$\bar{\partial}P F_{i,j}(P) = 0.$$

Then we have

$$\begin{aligned} F_{i,j+1}(P) &= \bar{\partial}(F_{i,j}(P)) - (F_{i,j}(P))\bar{\partial} - \bar{\partial}(F_{i,j}(P)) \\ &= \bar{\partial}(F_{i,j}(P)P) - (F_{i,j}(P))\bar{\partial}P \\ &= \bar{\partial}(F_{i,j}(P))P. \end{aligned}$$

So we only need to prove the conclusion will hold for $\bar{\partial}(F_{i,j}(P))P$. For the sake of simplicity of expression, we will assume that

$$F_{i,j}(P) = (\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1}.$$

Then we have

$$\begin{aligned} \bar{\partial}(F_{i,j}(P))P &= (\bar{\partial}^{i_1+1} P \partial^{j_1} P + \bar{\partial}^{i_1} P \bar{\partial} \partial^{j_1} P)(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1-1} + \dots \\ &+ (\bar{\partial}^{i_1} P \partial^{j_1} P)^{r-1} (\bar{\partial}(\bar{\partial}^{i_1} \partial^{j_1} P))(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1-r} + \dots \\ &+ (\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1-1} (\bar{\partial}^{i_1+1} P \partial^{j_1} P + \bar{\partial}^{i_1} P \bar{\partial} \partial^{j_1} P). \end{aligned} \quad (2.6.24)$$

By (1.1.2), we have

$$\bar{\partial}\partial^{j_1}P = \partial^{j_1}P\bar{\partial}P - \bar{\partial}P\partial^{j_1}P - \sum_{k=1}^{j_1-1} C_{j_1}^k (\partial^{j_1-k}P\bar{\partial}P\partial^kP).$$

And if $1 < r < l$, then we have

$$\begin{aligned} & (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}\bar{\partial}(\bar{\partial}^{i_1}\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &= (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1+1}P\partial^{j_1}P + \bar{\partial}^{i_1}P\bar{\partial}\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &= (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1+1}P\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &\quad + (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1}P\bar{\partial}\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &= (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1+1}P\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &\quad + (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1}P(\partial^{j_1}P\bar{\partial}P - \bar{\partial}P\partial^{j_1}P - \sum_{k=1}^{j_1-1} C_{j_1}^k (\partial^{j_1-k}P\bar{\partial}P\partial^kP)))(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \end{aligned}$$

Since $\bar{\partial}^{i_1}P = P\bar{\partial}^{i_1}P$ and $\bar{\partial}PP = 0$, we have

$$\bar{\partial}^{i_1}P\partial^{j_1}P(\bar{\partial}^{i_1}P\partial^{j_1}P\bar{\partial}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} = \bar{\partial}^{i_1}P\partial^{j_1}P\bar{\partial}^{i_1}P\partial^{j_1}P\bar{\partial}PP(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} = 0.$$

Similarly, by $\bar{\partial}^{i_1}P\bar{\partial}P = 0$, we also can prove that

$$\bar{\partial}^{i_1}P\partial^{j_1}P(\bar{\partial}^{i_1}P\bar{\partial}P\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} = 0.$$

That means

$$\begin{aligned} & (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}\bar{\partial}(\bar{\partial}^{i_1}\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &= (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1+1}P\partial^{j_1}P)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \\ &\quad - \sum_{k=1}^{j_1-1} C_{j_1}^k (\bar{\partial}^{i_1}P\partial^{j_1}P)^{r-1}(\bar{\partial}^{i_1}P\partial^{j_1-k}P\bar{\partial}P\partial^kP)(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l-r} \end{aligned} \tag{2.6.25}$$

By (2.6.24) and (2.6.25), we can see that Claim 4 also holds for $F_{i,j+1}(P)$. On the other hand, we have that

$$\begin{aligned} F_{i,j}(P) &= (\bar{\partial}^{i_1}P\partial^{j_1}P)^{l_1}(\bar{\partial}^{i_2}P\partial^{j_2}P)^{l_2}\dots(\bar{\partial}^{i_t}P\partial^{j_t}P)^{l_t} \\ &= P(\bar{\partial}^{i_1}P\partial^{j_1}P)^{l_1}(\bar{\partial}^{i_2}P\partial^{j_2}P)^{l_2}\dots(\bar{\partial}^{i_t}P\partial^{j_t}P)^{l_t} \\ &= P(F_{i,j}(P)). \end{aligned}$$

Note that

$$F_{i,j}(P)\partial P = 0.$$

Then

$$\begin{aligned} F_{i+1,j}(P) &= \partial(F_{i,j}(P)) - \partial P(F_{i,j}P) - (F_{i,j}(P))\partial P \\ &= \partial P F_{i,j}(P) + P(\partial(F_{i,j}(P))) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))P \\ &= P(\partial(F_{i,j}(P))). \end{aligned}$$

Similarly, we also can prove that Claim 4 holds for $F_{i+1,j}(P)$. Thus, we finish the proof of Claim 4.

By the proof of Claim 4, we also get the induction formulae of $F_{i,j}(P)$, $i, j \leq k$ as the following:

$$F_{i+1,j}(P) = P(\partial(F_{i,j}(P))), F_{i,j+1}(P) = (\bar{\partial}(F_{i,j}(P)))P. \tag{2.6.26}$$

□

Remark 2.7. From the proof of Lemma 2.6 and (2.6.26), we can see that the curvature formulae $F_{i,j}(P)$ does not depend on the chose of P .

Lemma 2.8. Let $P, Q \in \mathcal{P}_n(\Omega, \mathcal{U}) \cap \mathcal{A}_n(\Omega, \mathcal{U})$. And there exist holomorphic functions $\alpha, \beta : \Omega \rightarrow l^2(\mathbb{N}, B)$ such that

$$P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda), Q(\lambda) = \beta(\lambda) \cdot (\beta^*(\lambda) \cdot \beta(\lambda))^{-1} \cdot \beta^*(\lambda), \forall \lambda \in \Omega.$$

Let $F_{i,j}(P), F_{i,j}(Q)$ $i, j = 0, 1, \dots, n$ be differential functions in \mathcal{U} construct in Lemma 2.6 according to P and Q respectively.

Let $k \geq 1$ be an integer. Then there exists a unitary v such that

$$v \bar{\partial}^i P(\lambda) \partial^i P(\lambda) v^* = \bar{\partial}^j Q(\lambda) \partial^j Q(\lambda), \forall i, j \leq k$$

if and only if for any $\lambda \in \Omega$,

$$v F_{i,j}(P)(\lambda) v^* = F_{i,j}(Q)(\lambda), \forall i, j \leq k.$$

Proof. By Claim 4 of Lemma 2.6, each $F_{i,j}(P)$ may be expressed by as a sum of monomials of the form

$$(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t}.$$

Firstly, we have the following claim:

Claim 1 In the expression formulae of $F_{i,j}(P)$, $\bar{\partial}^i P \partial^j P$ appears only once.

In fact, when $i, j \leq 1$, we have

$$F_{1,1}(P) = \bar{\partial} P \partial P, F_{2,1}(P) = \bar{\partial}^2 P \partial P, F_{1,2}(P) = \bar{\partial} P \partial^2 P, F_{2,2}(P) = \bar{\partial}^2 P \partial^2 P - 2(\bar{\partial} P \partial P)^2.$$

If we assume that the Claim 1 holds for $i, j \leq k, k \geq 1$, then we only need to prove that Claim 1 will also hold for $i, j = k + 1$.

Since

$$\begin{aligned} F_{i+1,j}(P) &= \partial(F_{i,j}(P)) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P \\ &= \partial(\bar{\partial}^i \partial^j P + F_{i,j}(P) - \bar{\partial}^i \partial^j P) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P \\ &= \bar{\partial}^{i+1} P \partial^j(P) + \bar{\partial}^i P \bar{\partial} \partial^{j+1} P + \partial(F_{i,j}(P) - \bar{\partial}^i \partial^j P) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P \end{aligned}$$

If $\bar{\partial}^k P \partial^l P$ appears in the expression formulae of $\partial(F_{i,j}(P) - \bar{\partial}^i \partial^j P) - \partial P(F_{i,j}(P)) - (F_{i,j}(P))\partial P$, then $k < i$. So we can see that $\bar{\partial}^{i+1} P \partial^j(P)$ appears only once in the expression formulae of $F_{i+1,j}(P)$. Then we finish the proof of Claim 1.

Claim 2 Let v be a unitary of \mathcal{U} . Then

$$v F_{i,j}(P) v^* = F_{i,j}(Q), \forall i, j \leq k \Rightarrow v \bar{\partial}^i P \partial^j P v^* = \bar{\partial}^i Q \partial^j Q, \forall i, j \leq k.$$

In fact, when $i, j \leq 2$, we have

$$F_{1,1}(P) = \bar{\partial} P \partial P, F_{2,1}(P) = \bar{\partial}^2 P \partial P, F_{1,2}(P) = \bar{\partial} P \partial^2 P, F_{2,2}(P) = \bar{\partial}^2 P \partial^2 P - 2(\bar{\partial} P \partial P)^2.$$

If there exists unitary v such that

$$v F_{i,j}(P) v^* = F_{i,j}(Q), i, j \leq 2,$$

then

$$\begin{aligned} v(\bar{\partial} P \partial P) v^* &= v(F_{1,1}(P)) v^* = F_{1,1}(Q) = \bar{\partial} Q \partial Q, \\ v(\bar{\partial}^2 P \partial P) v^* &= v(F_{2,1}(P)) v^* = F_{2,1}(Q) = \bar{\partial}^2 Q \partial Q, \\ v(\bar{\partial} P \partial^2 P) v^* &= v(F_{1,2}(P)) v^* = F_{1,2}(Q) = \bar{\partial} Q \partial^2 Q, \end{aligned}$$

and

$$\begin{aligned} v(F_{2,2}(P))v^* &= v(\bar{\partial}^2 P \partial^2 P)v^* - 2v(\bar{\partial} P \partial P)^2 v^* \\ &= v(\bar{\partial}^2 P \partial^2 P)v^* - 2v(\bar{\partial} P \partial P)v^* v(\bar{\partial} P \partial P)v^* \\ &= \bar{\partial}^2 Q \partial^2 Q - 2(\bar{\partial} Q \partial Q)^2. \end{aligned}$$

It follows that

$$v(\bar{\partial}^2 P \partial^2 P)v^* = \bar{\partial}^2 Q \partial^2 Q.$$

If we assume the Claim 2 holds for the case of “ $k \leq l$ ”, then we only need to prove the conclusion holds for the case of $k = l + 1$.

Note that $F_{i,j}(P)$ may be expressed by as a sum of monomials of the form

$$(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t}.$$

And $\bar{\partial}^i P \partial^j P$ appears only once in the expression formulae of $F_{i,j}(P)$. Let $F_{i,j}(P) \sim F_{i,j}(Q)$ i.e. there exists unitary v such that

$$vF_{i,j}(P)v^* = F_{i,j}(Q), i, j \leq l.$$

By induction proof, we have that

$$v\bar{\partial}^i P \partial^j P v^* = \bar{\partial}^i Q \partial^j Q, i, j \leq l. \quad (2.8.1)$$

And if

$$vF_{l+1,l}(P)v^* = F_{l+1,l}(Q),$$

then we have

$$\begin{aligned} v(F_{l+1,l}(P))v^* &= v(\bar{\partial}^{l+1} P \partial^l P)v^* + v(F_{l+1,l}(P) - \bar{\partial}^{l+1} P \partial^l P)v^* \\ &= F_{l+1,l}(Q) \\ &= \bar{\partial}^{l+1} Q \partial^l Q + F_{l+1,l}(Q) - \bar{\partial}^{l+1} Q \partial^l Q \end{aligned}$$

Since $F_{l+1,l}(P) - \bar{\partial}^{l+1} P \partial^l P$ may be expressed by as a sum of monomials of the form

$$(\bar{\partial}^{\tilde{i}_1} P \partial^{\tilde{j}_1} P)^{l_1} (\bar{\partial}^{\tilde{i}_2} P \partial^{\tilde{j}_2} P)^{l_2} \dots (\bar{\partial}^{\tilde{i}_t} P \partial^{\tilde{j}_t} P)^{l_t}.$$

Since $\bar{\partial}^{l+1} P \partial^l P$ appears only once in the expression formulae of $F_{l+1,l}(P)$, we have

$$\tilde{i}_r, \tilde{j}_r \leq l, r \leq \tilde{t}.$$

By formulae (2.8.1), we have

$$v(F_{l+1,l}(P) - \bar{\partial}^{l+1} P \partial^l P)v^* = F_{l+1,l}(Q) - \bar{\partial}^{l+1} Q \partial^l Q.$$

So we have

$$v(\bar{\partial}^{l+1} P \partial^l P)v^* = \bar{\partial}^{l+1} Q \partial^l Q.$$

Similarly, we can prove that

$$v(\bar{\partial}^l P \partial^{l+1} P)v^* = \bar{\partial}^l Q \partial^{l+1} Q,$$

and

$$v(\bar{\partial}^{l+1} P \partial^{l+1} P)v^* = \bar{\partial}^{l+1} Q \partial^{l+1} Q.$$

Then we finish the proof of Claim 2.

Claim 3 Let v be a unitary of \mathcal{U} . Then

$$vF_{i,j}(P)v^* = F_{i,j}(Q), \forall i, j \leq k \Leftarrow v\bar{\partial}^i P \partial^j P v^* = \bar{\partial}^i Q \partial^j Q, \forall i, j \leq k.$$

Suppose that $F_{i,j}(P)$ is expressed by as a sum of monomials of the form

$$(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t},$$

and $i_r, j_r \leq k, r \leq t$.

Then we have

$$\bar{\partial}^{i_r} Q \partial^{j_r} Q = v \bar{\partial}^{i_r} P \partial^{j_r} P v^*, r \leq t,$$

and

$$v(\bar{\partial}^{i_1} P \partial^{j_1} P)^{l_1} (\bar{\partial}^{i_2} P \partial^{j_2} P)^{l_2} \dots (\bar{\partial}^{i_t} P \partial^{j_t} P)^{l_t} v^* = (\bar{\partial}^{i_1} Q \partial^{j_1} Q)^{l_1} (\bar{\partial}^{i_2} Q \partial^{j_2} Q)^{l_2} \dots (\bar{\partial}^{i_t} Q \partial^{j_t} Q)^{l_t}.$$

Then we finish the proof of Lemma 2.8. \square

Theorem 2.9. *Let $P, Q \in \mathcal{P}_n(\Omega, \mathcal{U}) \cap \mathcal{A}_n(\Omega, \mathcal{U})$. And there exist holomorphic functions $\alpha, \beta : \Omega \rightarrow l^2(\mathbb{N}, B)$ such that*

$$P(\lambda) = \alpha(\lambda) \cdot (\alpha^*(\lambda) \cdot \alpha(\lambda))^{-1} \cdot \alpha^*(\lambda), Q(\lambda) = \beta(\lambda) \cdot (\beta^*(\lambda) \cdot \beta(\lambda))^{-1} \cdot \beta^*(\lambda), \forall \lambda \in \Omega.$$

Then we have the following conclusions:

(1) *There exists $F_{i,j}(P), F_{i,j}(Q)$ $i, j = 0, 1, \dots, n$ which are linear combinations of $\bar{\partial}^{j_1} P \partial^{i_1} P \dots \bar{\partial}^{j_k} P \partial^{i_k} P$ and $\bar{\partial}^{j_1} Q \partial^{i_1} Q \dots \bar{\partial}^{j_k} Q \partial^{i_k} Q$ respectively such that*

$$F_{i,j}(P) = \alpha(-K_{P,z^i,\bar{z}^j}) h_1^{-1} \alpha^*, F_{i,j}(Q) = \alpha(-K_{Q,z^i,\bar{z}^j}) h_2^{-1} \alpha^*,$$

where $h_1 = \alpha^ \cdot \alpha, h_2 = \beta^* \cdot \beta$.*

(2) *$P \sim_u Q$ if and only if $F_{i,j}(P)(\lambda) \sim_u F_{i,j}(Q)(\lambda), \forall \lambda \in \Omega$, and $i, j = 0, 1, \dots, n-1$.*

Proof. By Lemma 1.6, we have $P \sim_u Q$ if and only if for each $\lambda \in \Omega$, there exists a unitary v such that

$$v \bar{\partial}^i P(\lambda) \partial^j P(\lambda) v^* = \bar{\partial}^i Q(\lambda) \partial^j Q(\lambda), \forall i, j \leq k.$$

By lemma 2.8, we have

$$v \bar{\partial}^i P(\lambda) \partial^i P(\lambda) v^* = \bar{\partial}^i Q(\lambda) \partial^i Q(\lambda), \forall i, j \leq k$$

if and only if for any $\lambda \in \Omega$,

$$v F_{i,j}(P)(\lambda) v^* = F_{i,j}(Q)(\lambda), \forall i, j \leq k.$$

Then we finish the proof of Theorem 2.9. \square

3. SIMILARITY OF COWEN-DOUGLAS OPERATORS AND THE CURVATURE FORMULAE

Theorem 3.1. *Let $T_1, T_2 \in B_n(\Omega) \cap SI$. Let α_1 and h_1 according to T_1 be given by Definition 3.2. Then $T_1 \sim T_2$ if and only if for any non-trivial idempotent $p \in \{T_1 \oplus T_2\}'$ there exists an idempotent $q \in \{T_1 \oplus T_2\}'$ which is murray-von Neumann equivalent to p and the fixed $F_{i,j}(q)$, $i, j = 0, 1, \dots, n$ and unitary v such that*

$$F_{i,j}(q)(\lambda) = v \alpha_1(\lambda) (-K_{T_1,z^i,\bar{z}^j}) h_1^{-1} \alpha_1^*(\lambda) v^*, \forall \lambda \in \Omega, \text{ and } i, j = 0, 1, \dots, n-1.$$

Let's recall the definition of Cowen-Douglas operator:

Definition 3.2. [6] Let Ω be a bounded and connected open subset of the complex plane \mathbb{C} and n a positive integer. Let $\mathcal{B}_n(\Omega)$ denote the set of operators T in $\mathcal{L}(\mathcal{H})$ satisfying:

- (1) $\Omega \subset \sigma(T) := \{\lambda \in \mathbb{C}, T - \lambda \text{ is not invertible}\};$
- (2) $\text{Ran}(T - \lambda) = \mathcal{H}$ for every $\lambda \in \Omega;$
- (3) $\bigvee_{\lambda \in \Omega} \{\ker(T - \lambda) : \lambda \in \Omega\} = \mathcal{H};$ and
- (4) $\dim \ker(T - \lambda) = n$ for every $\lambda \in \Omega.$

We call an operator in $\mathcal{B}_n(\Omega)$ a Cowen-Douglas operator with index n .

Let T be an operator in $\mathcal{B}_n(\Omega)$. By Example 2.3, we set $E(\lambda) = \text{Ker}(T - \lambda)$ and

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$$

where $\{\alpha_i(\lambda)\}_{i=1}^n$ are the frames of $E(\lambda)$ for any $\lambda \in \Omega$. And

$$h(\lambda) = \langle \alpha(\lambda), \alpha(\lambda) \rangle = \alpha^*(\lambda) \cdot \alpha(\lambda)$$

is the metric of E_T induced by α .

Following M. I. Cowen and R. G. Douglas, a curvature function for $T \in \mathcal{B}_n(\Omega)$ can be defined as:

Definition 3.3. [6]

$$K_T(\lambda) = -\frac{\partial}{\partial \bar{\lambda}}(h^{-1} \frac{\partial h}{\partial \lambda}), \text{ for all } \lambda \in \Omega,$$

where the metric

$$h(\lambda) = (\langle e_j(\lambda), e_i(\lambda) \rangle)_{n \times n}, \forall \lambda \in \Omega,$$

and $\{e_1(\lambda), e_2(\lambda), \dots, e_n(\lambda)\}$ are the frames of E_T . The partial derivatives of curvature are defined as the following:

Let E_T be a Hermitian holomorphic bundle induced by a Cowen-Douglas operator T , and K_T be a curvature of T . Then we have that

- (1) $K_{T, \bar{z}} = \frac{\partial}{\partial \bar{\lambda}}(K_T)$;
- (2) $K_{T, z} = \frac{\partial}{\partial \lambda}(K_T) + [h^{-1} \frac{\partial}{\partial \lambda} h, K_T]$.

By the definition above, we can get the partial derivatives of curvature: $K_{T, \lambda^i \bar{\lambda}^j}$, $i, j \in \mathbb{N} \cup \{0\}$ by using the inductive formulae above.

In order to study the similarity classification of Cowen-Douglas operators, we would like to introduce “strongly irreducible operator” first.

Definition 3.4. [10] An operator T is called strongly irreducible (denoted by “SI”), if there exists no non-trivial idempotent p ($p^2 = p$) in commutant of T (denoted by $\{T\}'$). Obviously, strongly irreducible is a kind of similarity invariant.

Cowen-Douglas operators of index 1 and unicellular operators are classical strongly irreducible operators. It is easy to observe that the adjoint of a strongly irreducible operator is still strongly irreducible and the spectrum of every strongly irreducible operator is connected.

In the following, we will give the proof of Theorem 3.1.

Proof. “ \implies ” Suppose that $T = T_1 \oplus T_2 \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ and $T_1 \sim T_2$, by Theorem in [10], we have the following statements:

- (1) $T_1 \oplus T_2 \sim_s T_1^{(2)} \sim_s T_2^{(2)}$;
- (2) $(K_0(\mathcal{A}'(T_1 \oplus T_2), \bigvee(\mathcal{A}'(T_1 \oplus T_2))), 1) \cong (\mathbb{Z}, \mathbb{N}, 1)$;

Let $p \in \mathcal{A}'(T_1 \oplus T_2)$ be a non-trivial idempotent. Let $[q]_0 = [1_{\mathcal{H}_1}]_0$. Then we have

$$T|_{\text{ran } p} \oplus T|_{\text{ran}(I-p)} \sim_s T|_{\text{ran } p} \oplus T|_{\text{ran}(I-p)} \sim_s T_1 \oplus T_2 \sim_s T_1^{(2)}.$$

By Main Theorem in [10], we have $T|_{\text{ran } p} \sim_s T|_{\text{ran } q}$ and $p \sim_s q \sim 1_{\mathcal{H}_1}$.

Let q be the holomorphic curve with $q(\lambda) := \{(\lambda, x) | x \in \text{Ker}(T|_{\text{ran } q} - \lambda)\}$, $\forall \lambda \in \Omega$.

Since $q \sim 1_{\mathcal{H}_1} \in \mathcal{A}'(T_1 \oplus T_2)$, we have $q(\lambda) \sim 1_{\mathcal{H}_1}(\lambda)$. By Theorem 2.9, there exists unitary v such that

$$F_{i,j}(q)(\lambda) = v F_{i,j}(1_{\mathcal{H}_1})(\lambda) v^* = v \alpha_1(\lambda) (-K_{T_1, z^i \bar{z}^j}) h_1^{-1} \alpha_1^*(\lambda) v^*, \forall \lambda \in \Omega, \text{ and } i, j = 0, 1, \dots, n-1.$$

“ \Leftarrow ” Choose $p = 1_{\mathcal{H}_2}$. Let $q(\lambda)$ be the holomorphic curve induced by q . If there exists an idempotent $q \in \{T_1 \oplus T_2\}'$ which is murray-von Neumann equivalent to $1_{\mathcal{H}_2}$. And there exists unitary v such that

$$F_{i,j}(q)(\lambda) = v\alpha_1(\lambda)(-K_{T_1, z^i, \bar{z}^j})h_1^{-1}\alpha_1^*(\lambda)v^*, \forall \lambda \in \Omega, \text{ and } i, j = 0, 1, \dots, n-1.$$

By Theorem 2.9, we have that $q(\lambda) \sim 1_{\mathcal{H}_1}(\lambda)$ and $T_2 \sim_s T|_{\text{ran } q} \sim T_1$.

□

4. TRACE OF THE DERIVATIVES OF CURVATURE

Recently, H.Kwon and S.Treil gave the following theorem to decide when a contraction operator T will be similar to the n times copies of S_z^* on Hardy space [13]. The following result only considers the case of Cowen-Douglas operators.

Theorem [H.Kwon and S. Treil] [14] *Let \mathbb{D} be the unit disk and $T \in B_n(\mathbb{D})$ with $\|T\| \leq 1$. For any $\lambda \in \mathbb{D}$, let $P(\lambda)$ be the orthogonal projection onto $\text{Ker}(T - \lambda)$. Then T is similar to the backward shift operator S_n^* if and only if there exists a bounded subharmonic function ψ such that*

$$\left\| \frac{\partial P(\lambda)}{\partial \lambda} \right\|_{HS}^2 - \frac{n}{(1 - |\lambda|^2)^2} \leq \Delta\psi(\lambda), \forall \lambda \in \mathbb{D},$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

And this result was also generalized to a general analytic functional space \mathcal{M}_n (see more details in [7]) by R. G. Douglas, H. Kwon and S. Treil. If we also only consider Cowen-Douglas operator class, then we have that

Theorem [R. G. DOUGLAS, H. KWON, and S. TREIL][7] *Let $T \in B_m(\mathbb{D})$ be an n -hypercontraction. Then T is similar to the backward shift operator S_{n, C^m}^* if and only if there exists a bounded subharmonic function ψ such that*

$$\left\| \frac{\partial P(\lambda)}{\partial \lambda} \right\|_{HS}^2 - \frac{nm}{(1 - |\lambda|^2)^2} = \Delta\psi(\lambda), \forall \lambda \in \mathbb{D}.$$

In [13], $\left\| \frac{\partial P(\lambda)}{\partial \lambda} \right\|_{HS}^2$ is pointed out to be the mean curvature of the eigenvector bundle $\text{Ker}(T - \lambda I)$ and Hardy shift case of this claim was also proved in [14]. And a proof of $B_1(\Omega)$ case was given by J. Sarkar in [26].

For any given Cowen-Douglas operator T , we can show that $\left\| \frac{\partial P}{\partial \lambda} \right\|_{HS}^2$ and curvature K_T have the following relationship:

Proposition 4.1 *Let $T \in B_n(\Omega)$ and $P : \Omega \rightarrow \mathcal{L}(\mathcal{H})$ be an extended holomorphic curve with $P(\lambda)$ is the projection induced by $\text{Ker}(T - \lambda)$ for any $\lambda \in \Omega$. Let $\{\sigma_i\}_{i=1}^\infty$ be the orthogonal normalize bases of \mathcal{H} . Then for any $s, t \leq n$, we have*

$$\sum_{i=1}^{\infty} \langle F_{s,t}(P)\sigma_i, \sigma_i \rangle = -\text{trace}(K_{T, z^s, \bar{z}^t}).$$

Proof. Let $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)^T$ be the coordinate of σ_i By Lemma 2.6, we have

$$F_{s,t}(P) = -\alpha K_{T, z^s, \bar{z}^t} h^{-1} \alpha^*, s, t \leq n.$$

Then it follows that for any i , we have

$$\begin{aligned}
\langle F_{s,t}(P)\sigma_i, \sigma_i \rangle &= -\langle \alpha K_{T,z^s,\bar{z}^t} h^{-1} \alpha^* e_i, e_i \rangle \\
&= -\langle K_{T,z^s,\bar{z}^t} h^{-1} \alpha^* e_i, \alpha^* e_i \rangle \\
&= -\langle K_{T,z^s,\bar{z}^t} h^{-1} \begin{pmatrix} \alpha_1^{1*} \alpha_1^{*2} \cdots \alpha_1^{l*} \cdots \\ \alpha_2^{1*} \alpha_2^{*2} \cdots \alpha_2^{l*} \cdots \\ \vdots \\ \alpha_n^{1*} \alpha_n^{*2} \cdots \alpha_n^{l*} \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}, \begin{pmatrix} \alpha_1^{1*} \alpha_1^{*2} \cdots \alpha_1^{l*} \cdots \\ \alpha_2^{1*} \alpha_2^{*2} \cdots \alpha_2^{l*} \cdots \\ \vdots \\ \alpha_n^{1*} \alpha_n^{*2} \cdots \alpha_n^{l*} \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix} \rangle \\
&= -\langle K_{T,z^s,\bar{z}^t} h^{-1} \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix}, \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix} \rangle
\end{aligned}$$

Now let

$$h = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{pmatrix}, h^{-1} = \begin{pmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1n} \\ \tilde{h}_{21} & \tilde{h}_{22} & \cdots & \tilde{h}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{h}_{n1} & \tilde{h}_{n2} & \cdots & \tilde{h}_{nn} \end{pmatrix},$$

and

$$-K_{T,z^s,\bar{z}^t} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix}.$$

Then we have

$$\begin{aligned}
& -\langle K_{T,z^s,\bar{z}^t} h^{-1} \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix}, \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix} \rangle \\
&= (\alpha_1^i, \alpha_2^i, \cdots, \alpha_n^i) K_{T,z^s,\bar{z}^t} h^{-1} \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix} \\
&= (\alpha_1^i, \alpha_2^i, \cdots, \alpha_n^i) \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{pmatrix} h^{-1} \begin{pmatrix} \alpha_1^{i*} \\ \alpha_2^{i*} \\ \vdots \\ \alpha_n^{i*} \end{pmatrix} \\
&= \left(\sum_{j=1}^n K_{j1} \alpha_j^i, \cdots, \sum_{j=1}^n K_{jn} \alpha_j^i \right) \begin{pmatrix} \sum_{k=1}^n \tilde{h}_{1,k} \alpha_k^{i*} \\ \sum_{k=1}^n \tilde{h}_{2,k} \alpha_k^{i*} \\ \vdots \\ \sum_{k=1}^n \tilde{h}_{n,k} \alpha_k^{i*} \end{pmatrix} \\
&= \left(\sum_{j=1}^n K_{j1} \alpha_j^i \right) \left(\sum_{k=1}^n \tilde{h}_{1,k} \alpha_k^{i*} \right) + \cdots + \left(\sum_{j=1}^n K_{jn} \alpha_j^i \right) \left(\sum_{k=1}^n \tilde{h}_{n,k} \alpha_k^{i*} \right) \\
&= \left(\sum_{j=1}^n \sum_{k=1}^n K_{j1} \tilde{h}_{1,k} \alpha_j^i \alpha_k^{i*} \right) + \cdots + \left(\sum_{j=1}^n \sum_{k=1}^n K_{jn} \tilde{h}_{n,k} \alpha_j^i \alpha_k^{i*} \right).
\end{aligned}$$

Thus, we have that

$$\sum_{i=1}^{\infty} \langle F_{s,t}(P)\sigma_i, \sigma_i \rangle = \left(\sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{k=1}^n K_{j1} \tilde{h}_{1,k} \alpha_j^i \alpha_k^{i*} \right) + \cdots + \left(\sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{k=1}^n K_{jn} \tilde{h}_{n,k} \alpha_j^i \alpha_k^{i*} \right) \quad (4.1.1)$$

Since $h^{-1}h = I$, then it follows that

$$\sum_{k=1}^n \tilde{h}_{l,k} h_{k,j} = 0, l \neq j;$$

and

$$\sum_{k=1}^n \tilde{h}_{l,k} h_{k,j} = 1, l = j.$$

Note that $h = \alpha\alpha^*$, then we have

$$h_{k,j} = \sum_{i=1}^{\infty} \alpha_j^i \alpha_k^{i*}.$$

Then we have

$$0 = \sum_{k=1}^n \tilde{h}_{l,k} h_{k,j} = \sum_{k=1}^n \tilde{h}_{l,k} \left(\sum_{i=1}^n \alpha_j^i \alpha_k^{i*} \right), j \neq l,$$

and

$$1 = \sum_{k=1}^n \tilde{h}_{l,k} h_{k,j} = \sum_{k=1}^n \tilde{h}_{l,k} \left(\sum_{i=1}^n \alpha_j^i \alpha_k^{i*} \right), j = l.$$

So for any $l = 0, 1, \dots, n$, we have

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^n \sum_{k=1}^n K_{j1} \tilde{h}_{l,k} \alpha_j^i \alpha_k^{i*} &= \sum_{j=1}^n K_{jl} \left(\sum_{k=1}^n \tilde{h}_{l,k} \sum_{i=1}^n \alpha_j^i \alpha_k^{i*} \right) \\ &= \sum_{j=1}^n K_{jl} \left(\sum_{k=1}^n \tilde{h}_{l,k} h_{k,j} \right) \\ &= K_{ll}. \end{aligned}$$

Thus, we have

$$\sum_{i=1}^{\infty} \langle F_{s,t}(P)\sigma_i, \sigma_i \rangle = \sum_{l=1}^n K_{ll} = -\text{trace}(K_{T,z^s, \bar{z}^t}).$$

□

By Proposition 4.1, we have the following corollary:

Corollary 4.2 For any operator $T \in B_n(\Omega)$, $\text{trace } K_T(\lambda) = -\|\partial P(\lambda)\|_{\mathcal{H}_S}^2$, $\forall \lambda \in \Omega$.

At last, inspired by R. G. Douglas, H. Kwon and S. Treil's similarity classification work, we would like to ask the following question:

Question 3 Can we use the trace of the partial derivatives of curvature (or $\sum_{i=1}^{\infty} \langle F_{s,t}(P)\sigma_i, \sigma_i \rangle$) to give a similarity classification theorem for the Cowen-Douglas operators?

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